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SIMILARITY OF DISTORTED RIVER MODELS WITH MOVABLE BED

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SIMILARITY OF DISTORTED RIVER MODELS WITH MOVABLE BED

H. A. Einstein,[1] M. ASCE, and Ning Chien,[2] A.M. ASCE

SYNOPSIS

The similarity conditions for distorted river models with movable bed are derived from the theoretical and empirical equations which have been found to describe the hydraulics and the sediment transport in such rivers. A complete numerical example is added to demonstrate the method of application to a particular river.

INTRODUCTION

There are many hydraulic engineering problems for which the basic equations are known but which are geometrically so complicated that the direct application of these equations becomes impossible. Many such problems can be solved today by the use of models which are shaped to duplicate the complicated geometry and in which the resulting flow patterns can be observed directly. Such a model permits the prediction of the corresponding prototype flows quantitatively only if the exact laws of model similarity are known. The prediction of the model scales cannot be based on simple dimensional considerations if the model is distorted and includes the motion of a movable bed. A different approach must be used to find compatible systems of model scales and distortions.

In 1944, the senior author of this paper published a short account(1) of how compatible systems of scales can be found. He proposed in that paper the derivation of model scales from empirical working equations rather than from the underlying differential equations of motion. It was pointed out there that equations must be used which are applicable in the same form both to model and prototype, and which should preferably have the form of power functions. The same approach will be used in this paper. Many new developments in the description of flow and sediment transport in alluvial rivers will be incorporated.

Before the conditions of similarity can be stated it is necessary to define the exact meaning of this term. Let us define that similarity may be said to exist

1) if to each point, time and process in one scale, which may be called prototype, a corresponding point, time and process of the other scale, which may be called the model, can be coordinated uniquely;
2) if the ratios of corresponding physical magnitudes between model and prototype are constant for each type of physical magnitude.

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This implies, for an undistorted model, that all magnitudes of equal dimension must follow the same scale ratio; also, the various laws which interrelate the variables, parameters and constants of the prototype must apply to the model, too. The results derived from the operation of such a hydraulic model may then be transferred to the prototype by the use of these scale ratios.

A distortion in the physical sense exists if there are two types of variables or parameters of equal dimension operative in the same problem which are physically sufficiently independent such that they can be given different scale ratios.

The application of distortions in engineering is not new. Many engineers apply daily distortions in the presentation of flat sections and profiles. The desirability of such distortion in hydraulic models has become apparent to everyone who has tried to design a model of wide and shallow watercourses in which the large horizontal dimensions call for a model scale which is much too small for application to the vertical direction. Such small water depth would very often cause laminar flow in the model which cannot be used to duplicate the turbulent prototype flow. Many laboratories have built extremely valuable models in which the vertical scale is much larger than the horizontal scale. It is impossible, however, to introduce one distortion alone. This becomes apparent by the fact that, for instance, the slopes are distorted in the same degree as the vertical heights. It is generally known that the water depths and the slope enter most of our friction formulas. The velocity scale must be chosen to satisfy the Froude condition and will satisfy the friction equation only if the roughness of the various boundaries is also properly distorted. We will see shortly that in the case of a sediment carrying stream still more distortions must be introduced to balance their effects in the various equations describing flow and sediment motion.

Nomenclature and Distortions

In the following discussion, the symbols with the subscripts, p and M, will indicate quantities, in prototype and model, respectively. The ratio of corresponding values in the two scales will be denoted by a subscript r. Thus, the ratio of depth scales, for instance, is defined as $h_r = h_p/h_M$; similarly, the following ratios are introduced:

L_r	ratio of horizontal lengths
h_r	ratio of vertical lengths
D_r	ratio of grain diameters
S_r	ratio of slopes, particularly energy slopes
V_r	ratio of horizontal flow velocities
$(\rho_s - \rho_f)_r$	ratio of sediment densities under water with ρ_f assumed to be equal in model and prototype
t_{1r}	ratio of hydraulic times
t_{2r}	ratio of flow durations (sedimentation time)
q_{Br}	ratio of bed-load rates (weight under water per unit of time and width)
q_{Tr}	ratio of total-load rates (weight under water per unit of time and width)
V_{sr}	ratio of sediment settling velocities
C_r	ratio of constants C (generalized Manning equation)
η_r	ratio of R_b'/R_T - values

The large list of variables and scales already implies that a number of distortions will be contemplated.

1) If L_r is independent from h_r, the model is vertically distorted.
2) If the grain size ratio D_r is different from both L_r and h_r, a third length scale is introduced and with it a second distortion.
3) If S_r is chosen independent from L_r and h_r, the model is assumed to be tilted in addition to the other distortions.
4) If the ratio of effective densities of the sediment $(\rho_s - \rho_f)_r$ is assumed to be different from the ratio of the fluid densities ρ_{fr} which is unity, then that represents a fourth distortion.
5) A fifth distortion is introduced if the time scale t_{1r} for the time values involved in the determination of velocities and sediment rates is chosen different from the ratio t_{2r} of durations for individual flow conditions, indicating the speed at which flow duration curves are duplicated.
6) A sixth distortion is introduced because of the impossibility to obtain suspended-load rates in a model at the same scale at which the bed-load rates are reproduced, making q_{Br} different from q_{Tr}.
7) A seventh and last distortion permits the ratio of settling velocities V_{sr} of corresponding grains to be different from the ratio of corresponding flow velocities.

Relationships Describing Alluvial Flows

It has already been stated that the flow and sediment description of the alluvial reaches which are to be studied by the model must be accomplished by identical formulas in model and prototype. These formulas can be transformed into relationships between the various ratios and thus represent conditions which must be satisfied when the various distortions are chosen. The individual relationships shall be discussed now using the various equations as they are given in Ref. 2. It is not believed that the choice of these equations in the exact form of Ref. 2 is a strong restriction to the generality of the method. Even if the reader prefers to substitute different formulas for the ones proposed here, he will find that he is not able to change the number of equations which must be satisfied in a particular problem. From this follows that he will have the same number of conditions and, therefore, the same number of degrees of freedom in choosing his ratios. He will find that his substitute equations will contain the same variables and that the entire difference will be a slightly changed set of exponents. These equations and the underlying criteria are

a. Friction Criterion: It has been pointed out in Refs. 2, 3 and 4 that the flow in an alluvial channel cannot be described generally by one formula with universal constants, but that it must be interpreted as a composite effect, the various parts of which follow different and independent laws. It can be shown easily that no distorted model is possible if the friction criterion is formulated as an identity, i.e., if one stipulates that both the grain resistance (surface drag) and the bar resistance (shape resistance) must both individually be similar. Actually this is not necessary for an overall similarity as long as the total friction behaves similarly. It is proposed, therefore, to base the frictional similarity on the behavior of the entire section for which it assumes the form of a rating curve. As most rating curves give approximately a straight line if the total discharge Q is plotted against the stages on a log-log sheet, it is assumed that the prototype and the model channel can be described by an equation of the form

$$V = \frac{c\sqrt{g}}{D^m} S^{1/2} h^{(1/2+m)} \tag{1}$$

which is a generalized Manning equation. Eq. (1) becomes identical with the Manning equation by taking m = 1/6 and by using the relationship $n \sim D^{1/6}$ (5). If it is assumed that the same exponent m can be used to describe both the prototype and the model relationships, at least for the most important range of discharges, the equation can be written between ratios as

$$V_r^2 S_r^{-1} h_r^{-1-2m} D_r^{2m} C_r^{-2} = \Delta_V \qquad \text{(A)}$$

The various ratios on the left side of eq. (A) are explained previously. The value Δ_V equals unity if the similarity is exactly satisfied, but may indicate a small deviation from the exact similarity if such a deviation becomes necessary for any practical reason.

While V_r, S_r, h_r and D_r are ratios which will recur in other equations, m and C_r are the exponent and the ratio of the constants in generalized eq. (1) in the form

$$V/\sqrt{R_T S g} = C(R_T/K_s)^m \qquad \text{(2)}$$

where R_T is the hydraulic radius of the total section with the bottom width as wetted perimeter and K_s the grain size of the bed representative for its grain roughness. If one may assume similar grain mixtures are used in model as exist in the prototype, the ratio of the K_s values equals that of the grain sizes D. The values C must be determined individually for model and prototype and the ratio found for a representative average channel section. Using the nomenclature of Ref. 2 to 4

$$\begin{aligned} R_T &= A_T/P_b = (A_b' + A_b'' + A_W)/P_b \\ &= (R_b' P_b + R_b'' P_b + R_W P_W)/P_b \\ &= R_b' + R_b'' + R_W P_W/P_b \end{aligned} \qquad \text{(3)}$$

The model values of C and m can be determined by a trial-and-error method only, as they depend on the choice of the remaining scale ratios. This procedure will be explained in the example of a "Big Sand Creek" model.

b. Froude Criterion: In open channel flows, Froude's law is one of the main criteria because it balances the gravitational forces against the inertia forces. This can easiest be expressed by the corresponding energies. Since the energy content of the flow may be divided into its kinetic and potential parts as velocity head and depth, both are correlated by the equation

$$V^2/2g + h = \text{Constant} \qquad \text{(4)}$$

Even if in river channels the water surface slope is often very small, one must remember that it is the gravitational force which maintains the flow. Furthermore, any river improvement plan usually involves certain types of river constriction work by the construction of training walls, jetties, and groins along the banks. At the constricted sections and especially in the neighborhood of the structures, the elevation of water surface may change rapidly. It is, therefore, advisable to include the Froude law as one of the criteria in designing river

models, as it safe-guards similar flow and energy loss around structures and other channel irregularities.

There appears to be a possibility that Froude criterion loses its significance in deep rivers when the Froude's number becomes very low ($F = V^2/gD << 1$). The only effect of velocity changes seems to be in that case an energy loss which is taken care of elsewhere. No experience exists on this part, however.

Froude law may be written in ratios as

$$V_r \, h_r^{-1/2} = \Delta_F \tag{B}$$

Here again stands $\Delta_F = 1$ for exact similarity while a deviation from unity measures a possible necessary deviation from the exact solution.

c. Sediment transport criterion: In order to have similar sediment transport conditions near the bed it is, in general, necessary that both ϕ_* and ψ_* are equal in model and prototype since the two are not connected by a power-type equation. Only if the transport rates are restricted to a very narrow range of values is it possible to combine the two conditions into one. The equality of the values

$$\phi_* = \frac{i_B}{i_b} \frac{q_B}{g(\rho_s - \rho_f)} \left(\frac{\rho_f}{\rho_s - \rho_f}\right)^{1/2} \left(\frac{1}{gD^3}\right)^{1/2} \tag{4}$$

is possible for all fractions of a mixture only if the two mixtures are similar, such that the ratios of the i-values become equal to unity. With ρ_f equal in model and prototype the equation of equal ϕ_*-values may be written as

$$q_{Br} (\rho_s - \rho_f)_r^{-3/2} \, D_r^{3/2} = 1 \tag{C}$$

No deviation from this equation is usually desired. The sediment rate q_B is measured in weight under water.

d. Zero Sediment-load Criterion: The equality of ψ_*-values in model and prototype is often interpreted by engineers as the condition of similar flow conditions at the beginning of sediment motion. It can be expressed as equal ψ_* values.

$$\psi_* = \left[\frac{\rho_s - \rho_f}{\rho_f} \frac{D}{R_b' S}\right] \xi Y \left(\frac{\log_{10} 10.6}{\log_{10} 10.6 X/\Delta}\right)^2 \tag{5}$$

In order to analyze the significance of all the corrections outside the brackets, let us remember that ξ is different for the various grain sizes of a mixture. For the larger sizes, both in model and prototype, ξ has the value 1. In order to have similarity the ratio of ξ-values must be unity for all sizes. Since ξ is a function of D/X, D_r must equal X_r. Referring again to Ref. 2, we find that

$$x = 1.39\delta \quad \text{for } \Delta/\delta < 1.8$$
$$= 0.77\Delta \quad \text{for } \Delta/\delta > 1.8 \qquad (6)$$

and
$$Y = Y(K_s/\delta)$$

with
$$\Delta = K_s/x \quad \text{and} \quad x = x(K_s/\delta)$$

and with $(K_s)_r = D_r$ by definition, it can be seen that in general similarity will be possible only if $\delta_r = D_r$. This condition will permit some slight deviation, however, since it controls some corrections only. With $\delta_r = D_r$ we see that $x_r = 1$ and that $\Delta_r = D_r$. Thus we can write

$$(\rho_s - \rho_f)_r D_r \eta_r^{-1} h_r^{-1} S_r^{-1} = 1 \qquad \text{(D)}$$

in ratios where the value η is defined by

$$\eta = R_b'/R_T \qquad (8)$$

or as the ratio of the hydraulic radius referring to the surface drag to the entire radius R_T. This correction must be introduced since $R_{Tr} = h_r$, but $R'_{br} \neq h_r$.

e. Laminar sublayer criterion: As has just been seen, δ_r must usually equal D_r to achieve similarity. This can be written in ratios as

$$D_r \eta_r^{1/2} h_r^{1/2} S_r^{1/2} = \Delta_\delta \qquad \text{(E)}$$

Again, a slight deviation Δ_δ may be permitted especially in cases where the bulk of the bed is considerably coarser than δ and, therefore, not directly affected by its value. In the derivation of eq (E) it may be noted that δ is dependent on the surface drag only, and thus must contain $u_*' = \sqrt{R_b' S g}$ which has a ratio $\eta_r^{1/2} h_r^{1/2} S_r^{1/2}$. The viscosity ν is assumed to be equal in model and prototype, but can be introduced differently if necessary.

f. By calculating for some characteristic flows in model and prototype the ratio q_T/q_B the average ratio of these ratios q_{Tr}/q_{Br} can be determined, it may be called B. This value can then be used to give a general relationship between the two load ratios

$$q_{Br} q_{Tr}^{-1} B = 1 \qquad \text{(F)}$$

Here in both q_B and q_T are measured in weight under water per unit of width and time.

g. Hydraulic time t_1 may be defined as the time which a water particle takes to move with velocity v through a distance L

$$V_r t_{1r} L_r^{-1} = 1 \qquad \text{(G)}$$

h. A different time is the time t_2 indicating the duration of individual flows.

This time ratio must be such that corresponding time intervals are required by corresponding sediment rates q_T to fill corresponding volumes. Expressed in ratios this equation can be written for the unit width as

$$q_{Tr}\, t_{2r}\, L_r^{-1}\, h_r^{-1}\, (\rho_s - \rho_f)_r^{-1} = 1 \qquad \text{(H)}$$

assuming the pore volume of the deposits to be equal in model and prototype. The sediment rates q_T are to be measured in weight under water. t_{2r} is the time scale at which the hydrographs must be repeated in the model.

i. Independent ratios have been introduced in this study for length, height and slope. This implies that the model is not only vertically distorted but in addition tilted. As the tilt is applied to the model during construction and as it is assumed to be proportional to the prototype slope, it can be applied to flows only which have at all points water surface and energy line slopes which are constant with time. This condition is not fulfilled wherever the flow reverses direction, such as under the influence of the tide or in most overbank flows. In all such cases no additional tilt can be permitted, which is expressed by

$$S_r\, L_r\, h_r^{-1} = \Delta_N \qquad \text{(I)}$$

$\Delta_N = 1$ represents zero tilt. A small tilt is represented by a small deviation of Δ_N from unity.

Solution of the Similarity Equations

Table 1 gives 9 equations (A) to (I) which are products of powers of the various ratios equal to unity. The numbers give the exponent of the various ratios.

There are 13 ratios, the exponent m and 4Δ - values involved. Of these are 10 free ratios. The exponent m and the three ratios must be determined by auxiliary calculations (as shown in the accompanying problem) while the Δ-values can be chosen.

Between the 10 free ratios there are 9 equations to be satisfied, such that only one of the 10 can be chosen freely if the Δ -values are held rigidly to unity or to some other fixed value. If some of the Δ -values are permitted to deviate from unity by a certain margin, it is possible to choose two or more of the free ratios within certain limits.

The nine equations are solved in Table II in three different ways: first, with the vertical scale h_r chosen freely, then with the horizontal scale L_r chosen, and finally, with the density of the model sediment chosen. One of the solutions will usually satisfy the particular need in designing such a model. The exact solution is obtained with all Δ -values equal to unity. If some deviations of Δ from unity are accepted, it is possible to choose more than only one scale arbitrarily. This is sometimes necessary when the size of the model is given by the available space, while the sediment density is determined by the availability of materials for the particular purpose. The design of a complete set of distorted model scales will now be demonstrated on a fictitious model of Big Sand Creek, the channel of which was used as an example for sediment load calculation in Ref. 2.

TABLE I

Model Laws for River Models with Sediment Motion

Eq.	L_r	h_r	V_r	S_r	D_r	$(\rho_s - \rho_f)_r$	q_{Br}	q_{Tr}	t_{1r}	t_{2r}	B^*	C_r^*	η_r^*	$\Delta_V^\bullet$	$\Delta_F^\bullet$	$\Delta_\delta^\bullet$	$\Delta_N^\bullet$	= 1 significance
A		-1-2m	2	-1	2m							-2		-1				Friction Eq.
B		-1	2												-2			Froude
C					-3	-3	2											ϕ
D		-1		-1	1	1							-1					ψ
E		1		1	2								1			-2		D/δ
F							1	-1			1							Suspended Load Ratio
G	-1		1						1									Definition of t_1
H	-1	-1				-1		1		1								Sediment continuity
I	1	-1		1													-1	No tilt

* B, C_r, η_r, and m are determined from auxiliary calculations.

• Δ - values may be chosen to suit the conditions.

TABLE II

Model Ratios for Open Channel Flows with Sediment Motion

Solution with known effects of leaving out eqs. (A), (B), (E) and (I)

(a) Choose h_r

Scale Ratio	Symbol	h_r	C_r	η_r	B	Δ_F	Δ_ζ	Δ_N	Δ_v
Horizontal lengths	L_r	$\frac{4m+1}{m+1}$	$\frac{2}{m+1}$	$\frac{m}{m+1}$	-	$\frac{-2}{m+1}$	$\frac{-2m}{m+1}$	1	$\frac{1}{m+1}$
Vertical lengths	h_r				chosen				
Flow velocity	V_r	$\frac{1}{2}$	-	-	-	1	-	-	-
Slope	S_r	$\frac{-3m}{m+1}$	$\frac{-2}{m+1}$	$\frac{-m}{m+1}$	-	$\frac{2}{m+1}$	$\frac{2m}{m+1}$	-	$\frac{-1}{m+1}$
Sediment size	D_r	$\frac{2m-1}{2(m+1)}$	$\frac{1}{m+1}$	$\frac{-1}{2(m+1)}$	-	$\frac{-1}{m+1}$	$\frac{1}{m+1}$	-	$\frac{1}{2(m+1)}$
Sediment density under water	$(\rho_s-\rho_f)_r$	$\frac{3(1-2m)}{2(m+1)}$	$\frac{-3}{m+1}$	$\frac{3}{2(m+1)}$	-	$\frac{3}{m+1}$	$\frac{2m-1}{m+1}$	-	$\frac{-3}{2(m+1)}$
Bed-load rate per unit width and time by wt. under water	q_{Br}	$\frac{3(1-2m)}{2(m+1)}$	$\frac{-3}{m+1}$	$\frac{3}{2(m+1)}$	-	$\frac{3}{m+1}$	$\frac{3m}{m+1}$	-	$\frac{-3}{2(m+1)}$
Total-load rate per unit width and time by wt. under water	q_{Tr}	$\frac{3(1-2m)}{2(m+1)}$	$\frac{-3}{m+1}$	$\frac{3}{2(m+1)}$	1	$\frac{3}{m+1}$	$\frac{3m}{m+1}$	-	$\frac{-3}{2(m+1)}$
Hydraulic time	t_{1r}	$\frac{7m+1}{2(m+1)}$	$\frac{2}{m+1}$	$\frac{m}{m+1}$	-	$\frac{-(m+3)}{m+1}$	$\frac{-2m}{m+1}$	1	$\frac{1}{m+1}$
Sedimentation time	t_{2r}	$\frac{5m+2}{m+1}$	$\frac{2}{m+1}$	$\frac{m}{m+1}$	-1	$\frac{-2}{m+1}$	$\frac{-(3m+1)}{m+1}$	1	$\frac{1}{m+1}$

TABLE II

Model Ratios for Open Channel Flows with Sediment Motion

Solution with known effects of leaving out eqs. (A), (B), (E) and (I).

(b) Choose L_r

Scale Ratio	Symbol	L_r	C_r	η_r	B	Δ_F	Δ_δ	Δ_N	Δ_v
Horizontal lengths	L_r	chosen							
Vertical lengths	h_r	$\frac{m+1}{4m+1}$	$\frac{-2}{4m+1}$	$\frac{-m}{4m+1}$	-	$\frac{2}{4m+1}$	$\frac{2m}{4m+1}$	$\frac{-(m+1)}{4m+1}$	$\frac{-1}{4m+1}$
Flow velocity	V_r	$\frac{m+1}{2(4m+1)}$	$\frac{-1}{4m+1}$	$\frac{-m}{2(4m+1)}$	-	$\frac{2(2m+1)}{4m+1}$	$\frac{m}{4m+1}$	$\frac{-(m+1)}{2(4m+1)}$	$\frac{-1}{2(4m+1)}$
Slope	S_r	$\frac{-3m}{4m+1}$	$\frac{-2}{4m+1}$	$\frac{-m}{4m+1}$	-	$\frac{2}{4m+1}$	$\frac{2m}{4m+1}$	$\frac{3m}{4m+1}$	$\frac{-1}{4m+1}$
Sediment size	D_r	$\frac{2m-1}{2(4m+1)}$	$\frac{2}{4m+1}$	$\frac{-(2m+1)}{2(4m+1)}$	-	$\frac{-2}{4m+1}$	$\frac{2m+1}{4m+1}$	$\frac{1-2m}{2(4m+1)}$	$\frac{1}{4m+1}$
Sediment density under water	$(\rho_s-\rho_f)_r$	$\frac{3(1-2m)}{2(4m+1)}$	$\frac{-6}{4m+1}$	$\frac{3(2m+1)}{2(4m+1)}$	-	$\frac{6}{4m+1}$	$\frac{2m-1}{4m+1}$	$\frac{3(2m-1)}{2(4m+1)}$	$\frac{-3}{4m+1}$
Bed-load rate per unit width and time by wt. under water	q_{Br}	$\frac{3(1-2m)}{2(4m+1)}$	$\frac{-6}{4m+1}$	$\frac{3(2m+1)}{2(4m+1)}$	-	$\frac{6}{4m+1}$	$\frac{6m}{4m+1}$	$\frac{3(2m-1)}{2(4m+1)}$	$\frac{-3}{4m+1}$
Total-load rate per unit width and time by wt. under water	q_{Tr}	$\frac{3(1-2m)}{2(4m+1)}$	$\frac{-6}{4m+1}$	$\frac{3(2m+1)}{2(4m+1)}$	1	$\frac{6}{4m+1}$	$\frac{6m}{4m+1}$	$\frac{3(2m-1)}{2(4m+1)}$	$\frac{-3}{4m+1}$
Hydraulic time	t_{1r}	$\frac{7m+1}{2(4m+1)}$	$\frac{1}{4m+1}$	$\frac{m}{2(4m+1)}$	-	$\frac{-2(2m+1)}{4m+1}$	$\frac{-m}{4m+1}$	$\frac{m+1}{2(4m+1)}$	$\frac{1}{2(4m+1)}$
Sedimentation time	t_{2r}	$\frac{5m+2}{4m+1}$	$\frac{-2}{4m+1}$	$\frac{-m}{4m+1}$	-1	$\frac{2}{4m+1}$	$\frac{-(2m+1)}{4m+1}$	$\frac{-(m+1)}{4m+1}$	$\frac{-1}{4m+1}$

TABLE II

Model Ratios for Open Channel Flows with Sediment Motion

Solution with known effects of leaving out eqs. (A), (B), (E) and (I)

(c) Choose $(\rho_s - \rho_f)_r$

Scale Ratio	Symbol	$(\rho_s - \rho_f)_r$	C_r	η_r	B	Δ_F	Δ_γ	Δ_N	Δ_v
Horizontal lengths	L_r	$\frac{-2(4m+1)}{3(2m-1)}$	$\frac{-4}{2m-1}$	$\frac{2m+1}{2m-1}$	-	$\frac{4}{2m-1}$	$\frac{2}{3}$	1	$\frac{-2}{2m-1}$
Vertical lengths	h_r	$\frac{-2(m+1)}{3(2m-1)}$	$\frac{-2}{2m-1}$	$\frac{1}{2m-1}$	-	$\frac{2}{2m-1}$	$\frac{2}{3}$	-	$\frac{-1}{2m-1}$
Flow velocity	V_r	$\frac{-(m+1)}{3(2m-1)}$	$\frac{-1}{2m-1}$	$\frac{1}{2(2m-1)}$	-	$\frac{2m}{2m-1}$	$\frac{1}{3}$	-	$\frac{-1}{2(2m-1)}$
Slope	S_r	$\frac{2m}{2m-1}$	$\frac{2}{2m-1}$	$\frac{-2m}{2m-1}$	-	$\frac{-2}{2m-1}$	-	-	$\frac{1}{2m-1}$
Sediment size	D_r	$\frac{-1}{3}$	-	-	-	-	$\frac{2}{3}$	-	-
Sediment density under water	$(\rho_s - \rho_f)_r$	chosen							
Bed-load rate per unit width and time by wt. under water	q_{Br}	1	-	-	-	-	1	-	-
Total-load rate per unit width and time by weight under water	q_{Tr}	1	-	-	1	-	1	-	-
Hydraulic time	t_{1r}	$\frac{-(7m+1)}{3(2m-1)}$	$\frac{-3}{2m-1}$	$\frac{4m+1}{2(2m-1)}$	-	$\frac{2(2-m)}{2m-1}$	$\frac{1}{3}$	1	$\frac{-3}{2(2m-1)}$
Sedimentation time	t_{2r}	$\frac{-2(5m+2)}{3(2m-1)}$	$\frac{-6}{2m-1}$	$\frac{2(m+1)}{2m-1}$	-1	$\frac{6}{2m-1}$	$\frac{1}{3}$	1	$\frac{-3}{2m-1}$

The Big Sand Creek Model

First, the friction conditions in the prototype are summarized in Table III based on Table 6 of Ref. 2. With S = 0.001050 and D_{65} = K_s = 0.00115 ft. the following values are obtained.

Table III

Friction Conditions - Big Sand Creek

R_b' ft	R_b'' ft	R_T ft	v ft/sec	$\eta_p = R_b'/R_T$	$\frac{R_T Sg}{v^2}$	R_T/K_s	$\frac{R_b' Sg}{v^2}$	R_b'/K_s	Q cfs.
0.50	0.86	1.30	2.92	0.384	0.00515	1130	0.00198	435	409
1.00	0.76	1.92	4.44	0.520	0.00330	1670	0.00172	870	1110
2.00	0.50	3.06	6.63	0.653	0.00236	2660	0.00154	1740	3710
3.00	0.30	4.40	8.40	0.682	0.00211	3820	0.00144	2610	9160
4.00	0.14	5.73	9.92	0.698	0.00197	4980	0.00137	3480	19850
5.00	0.07	7.04	11.30	0.711	0.00186	6110	0.00132	4350	35700

A similar hydraulic pattern for the model is now obtained by trial and error. First the length scale L_r = 150 is chosen on the basis of available room. Then the prototype friction is plotted in Fig. 1 as $\frac{v^2}{(R_T\ Sg)}$ against R_T/K_s (o-symbol) and the points for the higher discharges approximated by the line $A_{(1)}$-$A_{(1)}$ with the relationship

$$\frac{v^2}{R_T S g} = 21.05\left(\frac{R_T}{K_s}\right)^{0.372} \tag{9}$$

Next, the model resistance is estimated to follow the line $B_{(1)}$-$B_{(1)}$ with the equation

$$\frac{v^2}{R_T S g} = 30.76\left(\frac{R_T}{K_s}\right)^{0.372} \tag{10}$$

From these two relationships (9) and (10) one may find the values of C and m:

$$C_p = 4.595 \qquad C_M = 5.55 \qquad C_r = 0.827 \qquad m = 0.186$$

Assuming η_r = 1 and all Δ -values = 1,

$$
\left.
\begin{aligned}
h_r &= L_r^{\frac{m+1}{4m+1}} C_r^{\frac{-2}{4m+1}} \eta_r^{\frac{-m}{4m+1}} = 150^{0.68} \cdot 0.827^{-1.147} = 37.3 \\
S_r &= L_r^{\frac{-3m}{4m+1}} C_r^{\frac{-2}{4m+1}} \eta_r^{\frac{-m}{4m+1}} = 150^{-0.32} \cdot 0.827^{-1.147} = 0.25 \\
D_r &= L_r^{\frac{2m-1}{2(4m+1)}} C_r^{\frac{2}{4m+1}} \eta_r^{\frac{-(2m+1)}{2(4m+1)}} = 150^{-0.18} \cdot 0.827^{1.147} = 0.326 \\
(\rho_s - \rho_f)_r &= D_r^{-3} = 28.6
\end{aligned}
\right\} \qquad (11)
$$

From these ratios results the following model

$$S_M = \frac{0.00105}{0.25} = 0.0042 \qquad \rho_{s_M} = 1.059 \ gr/c.c.$$

$$D_{35_M} = 0.00288 \ ft. \qquad D_{65_M} = K_{s_M} = 0.00352 \ ft.$$

The model hydraulics of the first approximation is then given by Table IV. R_b' values used in Table IV correspond to those used in the prototype using $\eta_r h_r$ = 37.3.

Table IV

Model Hydraulics - First Approximation

R_b' ft	u_*' ft/s	δ ft	$\frac{D_{65}}{\delta}$	x	Δ ft	$\log_{10}(12.27 \frac{R_b'}{\Delta})$	v ft/s	ψ'	$\frac{v}{u_*}$
0.0134	0.0425	0.00289	1.22	1.59	0.00221	1.872	0.458	3.02	16.80
0.0268	0.0601	0.00204	1.72	1.46	0.00241	2.135	0.738	1.51	27.0
0.0536	0.0850	0.00145	2.44	1.27	0.00277	2.376	1.160	0.72	51.0
0.0805	0.1042	0.00118	2.98	1.18	0.00298	2.520	1.51	0.50	87.0
0.1072	0.1202	0.00102	3.44	1.14	0.00309	2.630	1.82	0.38	150
0.1342	0.1348	0.00091	3.86	1.10	0.00320	2.712	2.10	0.30	240

Table IV
(cont'd)

u_*'' ft/s	R_b'' ft	R_b ft	stage	R_T ft	$\frac{R_T S g}{v^2}$	$\frac{R_T}{K_s}$	$R_b'/R_T = \eta_M$	η_p	η_r
0.0273	0.0055	0.0189	4.010	0.020	0.0129	5.68	0.670	0.384	0.573
0.0273	0.0055	0.0323	4.034	0.0375	0.0093	10.64	0.714	0.520	0.730
0.0227	0.0038	0.0574	4.086	0.075	0.0076	21.3	0.716	0.653	0.911
0.0174	0.0022	0.0827	4.172	0.1240	0.0074	35.2	0.650	0.682	1.050
0.0121	0.0011	0.1083	4.260	0.1650	0.0068	46.4	0.650	0.698	1.073
0.0087	0.0006	0.1348	4.340	0.2000	0.0061	56.8	0.672	0.711	1.060

The values of the wall friction, of stage and R_T are determined by a trial and error method given in Table V and figure 2. First, the model bank roughness must be chosen. With the prototype banks rather rough, it is assumed that the Manning formula applies in model and prototype. The wetted perimeter of the banks was assumed in the prototype calculation (Ref. 2) to equal twice the water depth. In order to make the wall friction assume an equivalent part of the total friction in model and prototype, we write

$$A_{w_r} = R_{w_r} P_{w_r} = L_r h_r \tag{12}$$

from which we get for $P_{w_r} = h_r$ that $R_{w_r} = L_r$ (13)

Table V

Determination of R_T by Trial and Error Method

R_b ft	R_w ft	Stage ft	P_b ft	P_w ft	P_bR_b ft^2	P_wR_w ft^2	A_t ft^2
0.0189	0.0332	(4.000 (4.050 (4.100	0.50 0.86 1.22	- 0.07 0.17	.00945 .0162 .02285	- .0023 .00565	0.0095 0.0185 0.0285
0.0323	0.0672	(4.000 (4.050 (4.100	0.50 0.86 1.22	- 0.07 0.17	.01615 .0278 .0394	- .0047 .0114	0.0162 0.0325 0.0508
0.0574	0.1340	(4.050 (4.100 (4.150	0.86 1.22 1.57	0.07 0.17 0.27	.0493 .0700 .0900	.0094 .0228 .0362	0.0587 0.0928 0.1262
0.0827	0.1985	(4.100 (4.150 (4.200	1.22 1.57 1.95	0.17 0.27 0.37	.1010 .1300 .1612	.0338 .0536 .0735	0.1348 0.1836 0.2347
0.1083	0.264	(4.200 (4.250 (4.300	1.95 2.36 2.85	0.37 0.47 0.57	.2110 .2560 .3085	.0978 .1240 .1508	0.3088 0.3800 0.4593
0.1348	0.326	(4.300 (4.350 (4.400	2.85 3.36 3.87	0.57 0.67 0.77	.3840 .4530 .5220	.186 .218 .251	0.570 0.671 0.773

In the Manning equation for the banks, $V_r = \frac{1}{n_{w_r}} R_{w_r}^{2/3} S_r^{1/2}$, we get n_{w_r} by introducing previously determined values and $\Delta = 1$:

$$n_{w_r} = \frac{1}{V_r} R_{w_r}^{2/3} S_r^{1/2} = h_r^{-1/2} L_r^{2/3} \left(\frac{h_r}{L_r}\right)^{1/2} = L_r^{1/6} \tag{14}$$

It can thus be calculated $n_{w_r} = \sqrt[6]{150} = 2.305$ and $n_{w_M} = \frac{0.050}{2.305} = 0.0217$ and

$$R_w = \left(\frac{V n_w}{1.486\, S^{1/2}}\right)^{3/2} = \frac{V^{3/2}}{9.34}$$

Fig. 2 gives first the curves of P_w, P_b, R_T and A_T as functions of the stage, all derived from corresponding prototype curves and similarity scales (solid curves). For each calculated value of R_b Table V then determines values of

$A_T = R_b P_b + R_w P_w$ for various fictitious values of the stage. These A_T values for each R_b value are connected by a dashed line. The intersection with the solid A_T curve indicates the actual stage at which the R_b value occurs. These values of R_T are used in Table IV to determine η_r for the first approximation. Also $R_T Sg/V^2$ is calculated and plotted in Fig. 1 against R_T/K_s. It may be seen in this graph that the first assumed line $B_{(1)} - B_{(1)}$ was somewhat too high and too steep to describe the model points.

Thus a second approximation of a line $A_{(2)} - A_{(2)}$ for the prototype and parallel to it $B_{(2)} - B_{(2)}$ for the model was tried. The line $A_{(2)} - A_{(2)}$ does not cover the two smallest prototype discharges, is therefore not correct but at the larger flows. This is a minor defect, however, as most of the transport and of the bed changes occur at the medium and high flows.

The second approximation is based on $A_{(2)} - A_{(2)}$ for the prototype friction

$$\frac{V^2}{R_T S g} = 44.5 \left(\frac{R_T}{K_s} \right)^{0.290} \tag{15}$$

and on $B_{(2)} - B_{(2)}$ for the model friction

$$\frac{V^2}{R_T S g} = 52.6 \left(\frac{R_T}{K_s} \right)^{0.290} \tag{16}$$

From these one obtains

$$\left. \begin{array}{l} C_p = 6.66, \; C_M = 7.25, \; C_r = 0.902 \\ m = 0.145 \qquad \eta_r = 1.0 \quad \text{for } L_r = 150 \end{array} \right\} \tag{17}$$

The same formulas as quoted in the first approximation give

$$\left. \begin{array}{l} h_r = 42.25 \\ S_r = 0.282 \\ D_r = 0.292 \\ (\rho_s - \rho_f)_r = 40 \\ V_r = 6.5 \end{array} \right\} \tag{18}$$

This results in the following model dimensions

$$\left. \begin{array}{l} S_M = 0.00372 \\ D65_M = 0.00394 \text{ ft.} \\ D35_M = 0.00322 \text{ ft.} \\ \rho_{SM} = 1.041 \text{ gr./cc.} \end{array} \right\} \tag{19}$$

A new calculation of the hydraulics on this basis gave the (x) points of Fig. 1, which still follow with sufficient accuracy eq. (16). The results of eq. (18) and (19) are thus assumed to be usable together with

$$\eta_r = 1$$

$$L_r = 150$$

So far, the equations A, B, D, E and I have been used; eq. G gives

$$t_{1r} = L_r V_r^{-1} = 23.1$$

and eq. C gives

$$q_{Br} = D_r^{3/2} (\rho_s - \rho_f)_r^{3/2} = 40$$

The value of q_{Tr} is determined by eq. F which calls for the knowledge of the ratio B. These are determined in Table VI which lists first the calculated prototype total load $\Sigma\, i_T\, Q_T$ in tons per day, as determined in Table 8 of Ref. 1.

Table VI

Determination of ratios B

R'_{b_p} ft.	$(\Sigma i_T Q_T)_p$ tons/day	$(\Sigma i_T q_T)_p$ lbs. under water per sec. ft.	$(\Sigma i_T q_T)_M$ lbs. under water per sec. ft.,	$\frac{(\Sigma i_T q_T)_p / (\Sigma i_B q_B)_p}{(\Sigma i_T q_T)_M / (\Sigma i_B q_B)_M}$ = B	t_{2_r}	1 year duplicated in hours
0.5	700	0.9042	0.00173	1.36	4650	1.89
1.0	3790	0.420	0.00555	1.89	3350	2.62
2.0	33600	2.67	0.0184	3.63	1745	5.02
3.0	156000	9.11	0.0386	5.90	1075	8.15
4.0	538000	22.3	0.0685	8.15	778	11.28
5.0	1433000	46.2	0.1090	10.6	598	14.65

From a similar table the corresponding model values have been calculated and are given in the next column. The ratio of $(\Sigma i_B q_B)_p / (\Sigma i_B q_B)_M$ is equal to q_{Br} since corresponding i_B values are equal to prototype and model.

Eq. F and H permit now to determine t_{2r} as

$$t_{2r} = L_r h_r (\rho_s - \rho_f)_r\, q_{Br}^{-1} B^{-1} = \frac{6340}{B}$$

This time scale changes much with the stages and it appears necessary to extend the flood stages percentagewise over the low stages, a fact well known to the practical experimenter.

Significance of the Δ-Values

No use has been made so far of the Δ-values. Their significance may best be seen starting from the solution found as the second approximation. In this system a material is required for the model sediment of a specific gravity of 1.041. Let us assume no such material is available, but a material with the

specific gravity of 1.045 can be obtained. What would be the effect of such a deviation from the required values? This can easily be found by the choice of one of the Δ-values which appears to be least critical in the particular problem for the absorption of the deviations. The system of equations is solved again for the assumed values of L_r and $(\rho_s - \rho_f)_r$, introducing the chosen Δ as additional variable. One finds then that an entirely new system of scales will result together with the magnitude of the deviation which the chosen Δ-value must undergo to satisfy the remaining equations.

Reliability of the Method

The reader has by now probably received the impression that all the complications involved in the method have had no other effect but to make the system of similarities more and more unreliable. Such a statement is definitely unfair. But what the method does is to show that a distorted river model is in the best case an acceptable compromise which will permit the solution of certain problems which otherwise can not be solved except by experimentation in the prototype, which is under all conditions more expensive. This method of designing a model has one great advantage over all other such methods which are available at the present: It permits the prediction of its reliability at least in a qualitative way, with respect to the choice of Δ-values it gives such deviations even quantitatively.

What are the most important reasons for the loss of similarity?

1. In the description of the channel friction the modified Manning equation can not be expected to describe with equal m-values both model and prototype conditions over the entire range of discharges. It will be necessary, therefore, to permit deviations in the friction relationship for the less important flow conditions. This fact is most important if a large range of flow conditions exists in the problem area.

2. If flows must be considered in more than one channel, it becomes even more difficult to find a friction equation which describes all flows.

3. The total load rates must be used to determine the time ratio t_2 for flow durations. Since the bed-load rates must be made similar to obtain similar bed configurations and since the ratio between bed-load rates and total-load rates changes with the stage, a sliding time scale appears to become necessary in many cases, especially if the range of discharges is large. The determination of this scale becomes somewhat indeterminate if flow in more than one channel is important.

4. If the deposition of sediment in low velocity areas, such as overbank or in reservoirs, is an important phase of the problem under investigation, the wash load must be introduced into the model. Its characteristic is not given by the ability to be transported as bed load, however, but by its ability to stay in suspension permanently in the flow of the overbank areas.

Because of these difficulties it is absolutely necessary to verify any such model and its scales. Such a verification consists in the reproduction of a known prototype development in the model by similar flows. Only if such a verification is possible and successful can the model be depended upon for the prediction of future developments. It is common experience that the greatest difficulty of most model studies of this type is the gathering of the necessary prototype information for the construction, the operation, and particularly, the verification of the model. But it is felt that this method of designing model scales is able to shorten the time consuming trial and error method of finding the proper model scales so much that more effort can be spent on a reliable verification and the determination of the underlying river information.

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APPENDIX

List of Symbols

A_T	total area of a cross section
A_w	part of the cross section pertaining to the banks
A_b'	cross-sectional area pertaining to the grain
A_b''	cross-sectional area pertaining to irregularities
B	q_{Tr}/q_{Br}
C	constant in the generalized Manning's equation
D	grain size
D_{35}	grain size of which 35 percent is finer
D_{65}	grain size of which 65 percent is finer
g	gravitational acceleration
h	vertical lengths
i_b	fraction of bed material in a given grain size range
i_B	fraction of bed load in a given grain size range
i_T	fraction of total load in a given grain size range
K_s	roughness diameter
L	horizontal lengths
m	exponent in the generalized Manning's equation

n	friction factor in the Manning's equation
n_w	friction factor (Manning) of the banks
p_b	wetted perimeter of the bed
p_w	wetted perimeter of the banks
Q	flow discharge
q_B	bed-load rate in weight under water per unit of time and width
q_T	corresponding total-load rate
Q_T	total sediment load in cross section
R'_b	hydraulic radius with respect to the grain
R''_b	hydraulic radius for channel irregularities
R_T	hydraulic radius of the total section
R_w	hydraulic radius with respect to the bank
S	slope
t_1	hydraulic time
t_2	sedimentation time
u'_*	shear velocity with respect to the grain
u''_*	shear velocity for channel irregularities
V	horizontal flow velocity
V_s	settling velocity of a sediment particle
x	parameter for transition smooth - rough
X	characteristic grain size of a mixture
Y	pressure correction in transition smooth - rough
δ	the thickness of the laminar sub-layer
Δ	the apparent roughness diameter
Δ_a	deviation from the similarity law "a"
η	R'_b/R_T
ν	kinematic viscosity
ξ	"hiding factor" of grains in a mixture
ρ_f	density of the fluid
ρ_s	density of the solids
ϕ_*	intensity of transport for individual grain size
ψ'	intensity of shear on representative particle
ψ_*	intensity of shear for individual grain size

Subscripts

r	indicating ratio
p	refers to prototype
M	refers to model

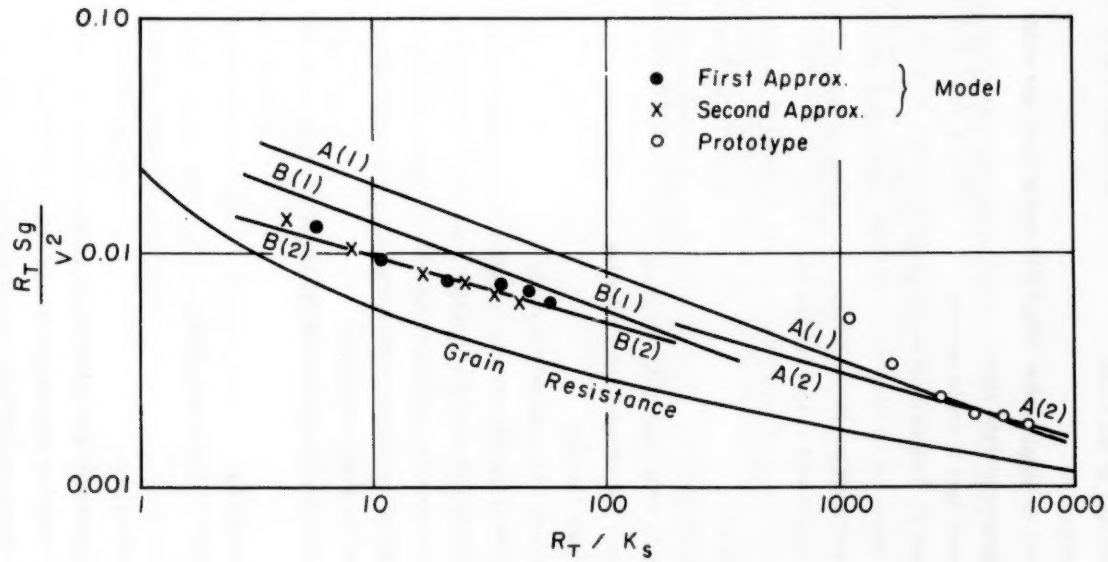

Fig. 1 Friction Condition For Big Sand Creek in Model and Prototype

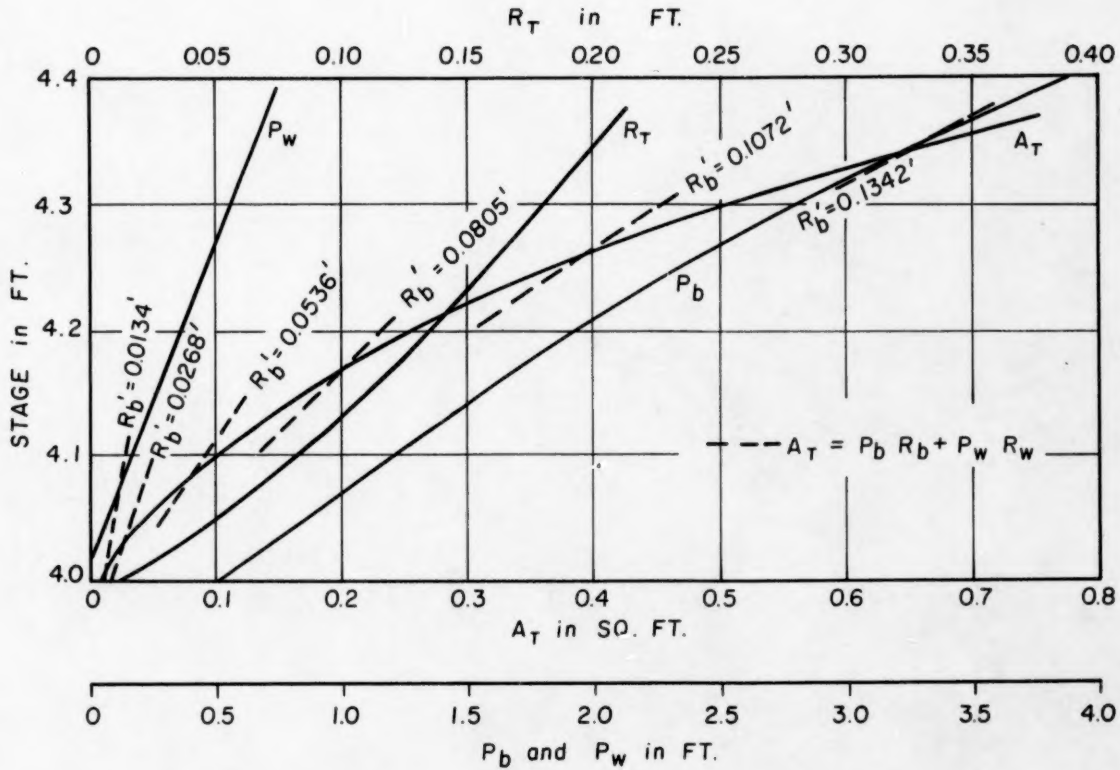

Fig. 2 Big Sand Creek Model-Trial and Error Solution For The Determination Of R_T

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